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Adaptive λ-tracking for polynomial minimum phase systems*

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Abstract. *It is proved that the adaptive controller*

$$u(t) = -k(t)\,\|e(t)\|^{s-1}\,e(t), \qquad e(t) := y(t) - y_{ref}(t) - n(t)$$

$$\dot{k}(t) = \begin{cases} (\|e(t)\| - \lambda)^r, & \|e(t)\| \geq \lambda \\ 0, & \|e(t)\| < \lambda \end{cases}$$

when applied to certain classes of multivariable nonlinear systems, achieves λ*-tracking, i.e. the trajectories of the closed-loop system are bounded and* $\|e(t)\| \to [0, \lambda]$ *as* $t \to \infty$. $\lambda > 0$ *and* $r \geq s$ *are design parameters, and* $s \geq 1$ *is some (known) upper bound of the polynomial degree of the right-hand side of the plant. The crucial assumptions on the system classes are minimum phase and strong relative degree-one. Classes encompass systems with sector bounded input nonlinearities, systems in input affine form, and systems not in input affine form but with bounded trajectories. The reference signals* $y_{ref}(\cdot)$ *and noise signals* $n(\cdot)$ *are only assumed to be absolutely continuous on bounded intervals and bounded with essentially bounded derivative. We also introduce modifications of the feedback strategy which preserve the simplicity of the controller but improve the transient behaviour.*

Nomenclature

$\mathbb{R}_+$ ($\mathbb{R}_-$)	the set of non-negative (non-positive) real numbers
$\mathbb{R}_+^*$	the set of positive real numbers
$\mathbb{C}_+$ ($\mathbb{C}_-$)	open right- (left-) half complex plane
$\sigma(A)$	the spectrum of the matrix $A \in \mathbb{C}^{n \times n}$
$\|x\|_P$	$= \sqrt{\langle x, Px \rangle}$ for $x \in \mathbb{R}^n$, $P = P^T \in \mathbb{R}^{n \times n}$ positive-definite
$\|x\|$	$= \|x\|_{I_n}$

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$\mathscr{B}_\lambda(0) \quad = \{e \in \mathbb{R}^m \mid \|e\| > \lambda\}$ for $m \in \mathbb{N}$, $\lambda > 0$

$L_p(I, \mathbb{R}^n)$ the vector space of measurable functions $f\colon I \to \mathbb{R}^n$, $I \subset \mathbb{R}$ an interval, such that $\|f(\cdot)\|_{L_p(I)} < \infty$, where

$$\|f(\cdot)\|_{L_p(I)} = \begin{cases} \left[\int_I \|f(s)\|^p \, ds \right]^{1/p} & \text{for} \quad p \in [1, \infty) \\ \operatorname*{ess\,sup}_{s \in I} \|f(s)\| & \text{for} \quad p = \infty \end{cases}$$

$\mathscr{W}^{1,\infty}$ the Sobolev space of functions $f\colon \mathbb{R}_+ \to \mathbb{R}^m$ which are absolutely continuous on compact intervals and $f(\cdot), \dot{f}(\cdot) \in L_\infty(\mathbb{R}_+, \mathbb{R}^m)$

For $\lambda, \rho > 0$ and positive-definite $P = P^T \in \mathbb{R}^{m \times m}$ we introduce, for $e \in \mathbb{R}^m$, the distance functions

$$d_\lambda(e) = \begin{cases} \|e\| - \lambda, & \|e\| \geq \lambda \\ 0, & \|e\| < \lambda \end{cases}$$

$$D_\rho(e) = \begin{cases} \|e\|_P - \rho, & \|e\|_P \geq \rho \\ 0, & \|e\|_P < \rho \end{cases}$$

$$\Theta_\rho(e) = \frac{D_\rho(e)}{\|e\|_P} e$$

1 Introduction

In the present paper, we extend a simple adaptive controller to various classes of nonlinear systems. The control objective is 'λ-tracking', i.e. for pre-specified $\lambda > 0$ the output $y(t)$ of a given system is supposed to track a reference signal $y_{\mathrm{ref}}(t)$, but we do not require asymptotic tracking, instead we guarantee that $y(t)$ approaches asymptotically the λ-neighbourhood of $y_{\mathrm{ref}}(t)$, i.e. $\lim_{t \to \infty} \operatorname{dist}(\|y(t) - y_{\mathrm{ref}}(t)\|, [0, \lambda]) = 0$. Since $\lambda > 0$ is pre-specified and might be arbitrarily small, the control objective is completely sufficient for applications.

We follow up a high-gain approach that is applicable to systems which can be stabilized by proportional output feedback. Since the system is unknown, the gain has to be found adaptively. This adaptive high-gain approach goes back to Morse (1983) and Willems and Byrnes (1984) who introduced the universal stabilizer $u(t) = -k(t)y(t)$, $\dot{k}(t) = y(t)^2$ for linear, single-input, single-output, minimum phase systems. This idea was extended to many more general classes and also to the asymptotic tracking problem with an internal model; see, for instance, Ilchmann (1991) for a survey. Miller and Davison (1991) introduced a 'dead-zone' but a more complicated stepwise adaptation strategy. The simple gain adaptation

$$\dot{k}(t) = \begin{cases} (\|e\|(t)\| - \lambda) \, \|e(t)\|, & \|e(t)\| \geq \lambda \\ 0, & \|e(t)\| < \lambda \end{cases} \tag{1}$$

with proportional time-varying output feedback

$$u(t) = -k(t)e(t), \qquad e(t) = y(t) - y_{\mathrm{ref}}(t) \tag{2}$$

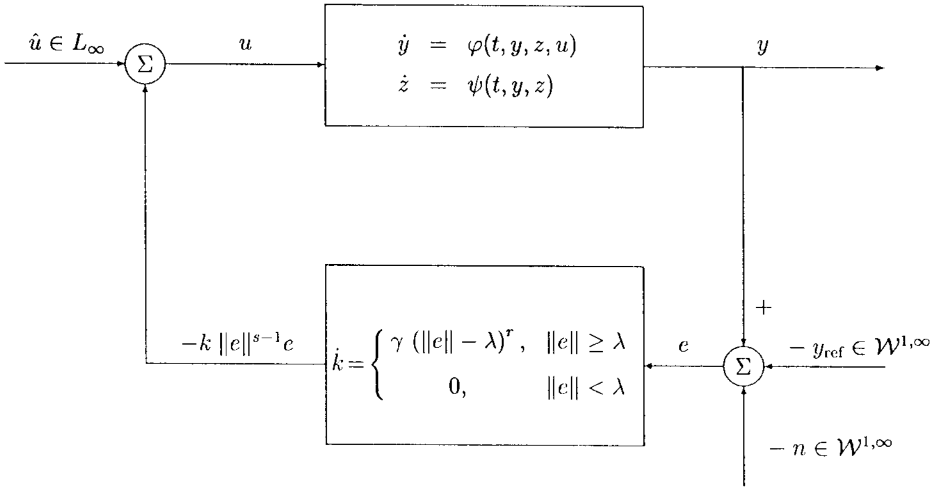

Fig. 1. Adaptive λ-tracking.

was introduced by Ilchmann and Ryan (1994) and extended by Allgöwer and Ilchmann (1995) and Allgöwer *et al.* (1997) to affine linearly bounded nonlinear systems.

Advantages of this approach are: the adaptation can cope with output corrupted noise, it is applicable to large classes of reference and noise signals whilst an internal model is not invoked, it is very simple in its design and does not invoke any identification mechanism or probing signals, the transient behaviour is very good compared to the few information available to the controller, it is applicable to numerous classes of nonlinear systems as long as they have global stable zero dynamics and strict relative degree-one.

This approach, and slight modifications thereof, was successful for many biotechnological applications: methanol synthesis in a polytropic, catalytic continuous stirred tank reactor on solid phase catalyst (Allgöwer *et al.*, 1997), control of a reaction in an exothermic continuous stirred tank reactor (Allgöwer & Ilchmann, 1995), control of anaerobic digestion by micro-organisms of animal wastes, e.g. chicken manure (Ilchmann & Weirig, 1996), and control of a Biogas Tower Reactor for the waste water treatment from baker's yeast production (Ilchmann & Pahl, 1998). This was not only tested by simulations but the multivariable high-gain λ-tracker worked successfully over several months on an industrial pilot reactor of full scale plant (20 m) at the DHW (Deutsche Hefewerke Hamburg, FRG).

In contrast with these precursors just mentioned, which deal rather with 'mild' nonlinear perturbations, the present paper considers strongly nonlinear systems but of relative degree-one. The nonlinearities are supposed to be polynomially bounded and only an upper bound of the degree of the polynomial bound needs to be known. The idea to use a feedback strategy of the form (2) is due to Ryan (1998). Ryan proves, apart from other results via discontinuous feedback, λ-stabilization for scalar systems and stabilization for nth order systems, stabilization for planar systems with unknown high-frequency gain by invoking a Nussbaum function (see Nussbaum, 1983). However, since in most applications the high-frequency gain is either known or can easily be detected, in the present paper we restrict our attention to cases where this is known.

There are several novelties of the present approach: it covers more classes of systems such as systems with polynomial nonlinearities, sector bounded input nonlinearities and certain nonlinear systems not in input affine form.

Equally important is that the proof is unifying and conceptually easier to understand. We extract the essential assumptions of the system class and separate them from the gain adaptation. This sets us in a position to give a rather simple proof which does not rely on LaSalle's Invariance Principle. If we apply this result to specific system classes, the difficulty lies in proving so-called 'high-gain lemmata'. However, this gives also a deeper structural insight into certain nonlinear systems.

Moreover, it enables us to choose the gain adaptation fairly generally. The gain adaptation (1) used by Ilchmann and Ryan (1994) is very specific and tailored for a Lyapunov function used in their proof. Intuitively, it was not clear why not to use

$$\dot{k}(t) = \begin{cases} (\| e \| (t) \| - \lambda)^r, & \| e(t) \| \geqslant \lambda \\ 0, & \| e(t) \| < \lambda \end{cases} \tag{3}$$

for $r \geqslant 1$, which is an extension of the well-known adaptive 'stabilizing' gain adaptation $\dot{k}(t) = \| y(t) \|^r$. This is also solved in the present paper.

Equation (3) is not only of theoretical interest but it improves the transient behaviour: the larger r is chosen the better the transient behaviour is. If $\| e(t) \| > \lambda$ and r is large, then the gain $k(t)$ increases fast and hence tracking is achieved faster. This is illustrated in Section 7.

We would like to finalize this introduction by illustrating the concept of high-gain λ-tracking for the simple example of scalar nonlinear systems of the form:

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \qquad y(t) = cx(t), \qquad x(0) = x_0 \in \mathbb{R} \tag{4}$$

where the only structural assumptions being made are: $c \in \mathbb{R}$ and the continuous functions $f(\cdot)$ and $g(\cdot)$ satisfy, for all $x \in \mathbb{R}$

$$\text{there exists some (unknown) } \sigma > 0 \text{ such that } \sigma < cg(x) \tag{5}$$

$$\text{there exists some (unknown) polynomial } p(\cdot) \text{ such that } |f(x)| < p(|x|) \tag{6}$$

Only the upper bound $s \geqslant \deg p(\cdot)$ needs to be known. We want to show that the time-varying and nonlinear feedback law

$$u(t) = -k(t) \, \| e(t) \|^{s-1} e(t), \qquad e(t) = y(t) - y_{\text{ref}}(t) \tag{7}$$

with the simple gain adaptation (3) for $r \geqslant s$ achieves λ-tracking (i.e. $|e(t)| \to [0, \lambda]$ as $t \to \infty$) for arbitrary initial conditions $k_0, x_0 \in \mathbb{R}$ and reference signals $y_{\text{ref}}(\cdot)$, as long as $y_{\text{ref}}(\cdot)$ and $\dot{y}_{\text{ref}}(\cdot)$ are essentially bounded.

Let $[0, \omega)$ denote the maximal interval of existence of the solution of the closed-loop system (4), (7), (3) for some $\omega \in (0, \infty]$.

We first prove $k(\cdot) \in L_\infty([0, \omega), \mathbb{R})$. Seeking a contradiction, suppose $\lim_{t \to \omega} k(t) = \infty$. (Note that $t \mapsto k(t)$ is monotonically non-decreasing by construction.) Differentiation of the Lyapunov like candidate $V_\lambda(e) := \frac{1}{2} d_\lambda(e)^2$ (see Nomenclature) along the solution of the closed-loop system (4), (7), (3) yields, for almost all $t \in [0, \omega)$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} V_\lambda(e(t)) &= d_\lambda(e) \frac{e}{|e|} [cf(x) - kcg(x) |e|^{s-1} e - \dot{y}_{\text{ref}}] \\ &\leqslant -k\sigma d_\lambda(e) |e|^s + d_\lambda(e) |cf(x) - \dot{y}_{\text{ref}}| \end{aligned} \tag{8}$$

By (6) and boundedness of $\dot{y}_{\text{ref}}$ and y_{ref}, there exist some $M>0$ such that $|cf(x)-\dot{y}_{\text{ref}}| \leqslant M[1+|e|^s]$, and hence

$$\frac{\mathrm{d}}{\mathrm{d}t} V_\lambda(e(t)) \leqslant -[k\sigma - M]\, d_\lambda(e)|e|^s + Md_\lambda(e)$$

Now we use the fact that

$$1 \leqslant \frac{1}{\lambda}|e| \leqslant \frac{1}{\lambda^s}|e|^s \qquad \text{if} \qquad d_\lambda(e) \geqslant 0 \tag{9}$$

to conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t} V_\lambda(e(t)) \leqslant -\left[k\sigma - M - \frac{M}{\lambda^s}\right] d_\lambda(e)|e|^s \tag{10}$$

Choose t_0 sufficiently large so that

$$k(t)\sigma - M - \frac{M}{\lambda^s} > 0 \qquad \text{for all} \qquad t \in [t_0, \omega)$$

Then substituting (9) into (10) yields, for all $t \in [t_0, \omega)$

$$\frac{\mathrm{d}}{\mathrm{d}t} V_\lambda(e(t)) \leqslant -2\lambda^s\left[k(t)\sigma - M - \frac{M}{\lambda^s}\right] V_\lambda(e(t))$$

and hence $V_\lambda(e(t))$ decays exponentially to zero. This contradicts, by the dead-zone in (3), unboundedness of $k(\cdot)$. Therefore, we have established that $k(\cdot)$ is bounded.

In a second step we prove that $\omega = \infty$. Seeking again a contradiction, suppose $\omega < \infty$. Then maximality of $[0,\omega)$ yields $\lim_{t\to\omega} |e(t)| = \infty$. Invoking (6), boundedness of $e(\cdot), y_{\text{ref}}(\cdot), \dot{y}_{\text{ref}}(\cdot)$ and $r \geqslant s$, yields, for some $M_1 > 0$ and almost all $t \in [0, \omega)$

$$\begin{aligned}
\dot{e}(t) &= cf(x(t)) - kcg(x(t))|e(t)|^{s-1}e(t) - \dot{y}_{\text{ref}}(t) \\
&\leqslant M_1[1+|e(t)|^s] \\
&\leqslant M_1[1+(\lambda + d_\lambda(e(t)))^s] \\
&\leqslant M_1[1+2^s(\lambda^s + d_\lambda(e(t))^s)] \\
&\leqslant M_1[1+(2\lambda)^s] + M_1 2^s d_\lambda(e(t))^s \\
&\leqslant M_1[1+(2\lambda)^s] + M_1 2^s[1+\dot{k}(t)]
\end{aligned}$$

Now, by integration, and since $k(\cdot)$ is bounded and ω is finite, we may conclude boundedness of $e(\cdot)$. This is a contradiction and hence the system does not have a finite escape time.

$k \in L_\infty(\mathbb{R}_+, \mathbb{R})$ is equivalent to $d_\lambda(e)^r \in L_1(\mathbb{R}_+, \mathbb{R})$. One can prove that

$$\frac{\mathrm{d}}{\mathrm{d}t} d_\lambda(e(t))^r \leqslant r d_\lambda(e(t))^{r-1}|\dot{e}(t)| \in L_\infty(\mathbb{R}_+, \mathbb{R})$$

and therefore $d_\lambda(e(t)) \to 0$ as $t \to \infty$. This proves $\lim_{t\to\infty} \operatorname{dist}(|e(t)|, [0,\lambda]) = 0$.

The paper is organized as follows. In Section 2 we introduce the general class of systems to be considered, the feedback law and the gain adaptation. Once one has found the appropriate concept respective assumptions, it is technically not too difficult to prove λ-tracking for a large class of systems. This will be done in Section

3. In Section 4 we then show that the abstract formulated class of systems covers nonlinear minimum phase systems with strong relative degree-one and, in Section 5, polynomial minimum phase systems with bounded input nonlinearities. In Section 6 certain minimum phase systems, occurring in the description of bio-reactors, are proved to be also suited for λ-tracking. These systems are not in input affine form. We finalize the paper with Section 7, where some simulations show the transient behaviour of the regulation and the advantages of the gain adaptation (3) when compared to (1).

2 System classes, adaptive regulator and control objectives

The 'class of systems' to be considered consists of multivariable, time-varying, nonlinear systems of the form

$$\boxed{\begin{array}{ll} \dot{y}(t) = \varphi(t, y(t), z(t), u(t)), & y(0) = y_0 \\ \dot{z}(t) = \psi(t, y(t), z(t)), & z(0) = z_0 \end{array}} \tag{11}$$

where, for $m, p \in \mathbb{N}$

$$\begin{array}{ll} \varphi: \mathbb{R}_+ \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^m \\ \psi: \mathbb{R}_+ \times \mathbb{R}^m \times \mathbb{R}^p \qquad\quad \to \mathbb{R}^p \end{array}$$

are assumed to be 'Carathéodory functions'* satisfying certain assumptions (A1)–(A4) to be specified later. The dimension p of the internal variable z need not be known.

As usual, $u(\cdot)$: $[0, \infty) \to \mathbb{R}^m$ denotes a locally integrable input vector and $y(\cdot)$ denotes the output to be controlled.

The 'class of reference signals' to be tracked as well as the 'class of noise signals' which may corrupt the output is $\mathcal{W}^{1,\infty}$, i.e. the Sobolev space of functions which are absolutely continuous on compact intervals and are essentially bounded with essentially bounded derivative

$$y_{\mathrm{ref}}(\cdot), n(\cdot) \in \mathbb{W}^{1,\infty}$$

The 'adaptive feedback law' is a simple time-varying and nonlinear error feedback of the form

$$\boxed{\begin{array}{l} e(t) = y(t) - y_{\mathrm{ref}}(t) - n(t) \\ u(t) = -k(t) \, \| e(t) \|^{s-1} e(t) + \hat{u}(t) \end{array}} \tag{12}$$

where $s \geqslant 1$, $y_{\mathrm{ref}}(\cdot), n(\cdot) \in \mathbb{W}^{1,\infty}$ and $\hat{u}(\cdot) \in L_\infty(\mathbb{R}_+, \mathbb{R}^m)$. s is, for many subclasses of (11), an upper bound of the polynomial degree of $\varphi(\cdot)$. $\hat{u}(\cdot)$ might have been appropriate in non-adaptive circumstances, but if no information is available to the designer he might set $\hat{u}(\cdot) \equiv 0$.

*α: $\mathbb{R} \times \mathbb{R}^q \to \mathbb{R}$ is called a Carathéodory function, if $\alpha(\cdot, x)$: $t \mapsto \alpha(t, x)$ is measurable on $\mathbb{R}$ for each $x \in \mathbb{R}^q$, and $\alpha(t, \cdot)$: $x \mapsto \alpha(t, x)$ is continuous on $\mathbb{R}^q$ for all $t \in \mathbb{R}$.

The scalar 'gain' is a monotonically non-decreasing function determined by the 'adaptive' law'

$$\boxed{\dot{k}(t) = \begin{cases} \gamma(\| e \| (t) \| - \lambda)^r, & \| e(t) \| \geqslant \lambda \\ 0, & \| e(t) \| < \lambda \end{cases}} \tag{13}$$

where arbitrary λ, $\gamma > 0$, $r \geqslant 1$ and $k(0) = k_0$ are design parameters free to choose. λ clearly specifies the desired performance with respect to the maximal allowable control tolerance. The parameter γ adjusts the speed of adaptation and a sensible choice lies in the order of magnitude of the inverse of the dominant time constant of the plant. r influences the dynamic of the adaptation and the larger r, the smaller the terminal gain $k_\infty := \lim_{t \to \infty} k(t)$. For polynomially bounded systems r has to be chosen greater or equal to the polynomial degree of the system in order to prevent finite escape time of the closed-loop system. The size of $k_0 \in \mathbb{R}$ should not be overestimated, as unnecessary large values only increase the sensitivity to measurement noise.

Note that $t \mapsto k(t)$ is a differentiable, non-decreasing function and that the gain adaptation incorporates a 'dead-zone' so that the gain is kept constant as soon as the error enters the ball around 0 of pre-specified radius $\lambda > 0$.

2.1 Assumptions on the plant

Let $y_{\text{ref}}(\cdot), n(\cdot) \in \mathcal{W}^{1,\infty}$, $\hat{u}(\cdot) \in L_\infty(\mathbb{R}_+, \mathbb{R}^m)$ and let, for some $\omega \in (0, \infty]$, $k(\cdot)$: $[0, \omega) \to \mathbb{R}$ be a continuous, non-decreasing function. If the feedback law (12) is applied to (11), then it is well known from the theory of ordinary differential equations, that the initial value problem (11), (12) admits a maximal solution on $[0, \tilde{\omega})$, for some $\tilde{\omega} \in (0, \omega]$, i.e. an absolutely continuous function

$$(y(\cdot), z(\cdot), k(\cdot)) \colon [0, \tilde{\omega}) \to \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R} \tag{14}$$

which satisfies (11), (12) almost everywhere on $[0, \tilde{\omega})$ and this interval is maximally extended. We assume that for the closed-loop system (11), (12) we have $\omega = \tilde{\omega}$ and that the following assumptions are satisfied:

(A1) High-gain property: if $k(\cdot) \notin L_\infty([0, \omega), \mathbb{R})$, then $\lim_{t \to \omega} e(t) = 0$.
(A2) Bounded internal variable: if $k(\cdot) \in L_\infty([0, \omega), \mathbb{R})$, then $z(\cdot) \in L_\infty([0, \omega), \mathbb{R}^p)$.
(A3) No finite escape time of $y(\cdot)$: if $\omega < \infty$, then $y(\cdot) \in L_\infty([0, \omega), \mathbb{R}^m)$.
(A4) Locally bounded derivative: if $k(\cdot) \in L_\infty([0, \omega), \mathbb{R})$, then for every $L > 0$ there exists some $L' > 0$ so that, for almost all $t \in [0, \omega)$

$$\| y(t) \| < L \;\Rightarrow\; \| \dot{y}(t) \| < L'$$

Assumption (A1) is crucial, since together with the 'dead-zone' gain adaptation (13) it yields the boundedness of $k(\cdot)$. This, together with (A2), gives existence of every solution of the closed-loop system (11), (12) on the whole of $[0, \infty)$. Finally, after having proved a technical lemma, it is easy to see that (A4) together with the gain adaptation (13) ensures λ-tracking.

2.2 Control objectives

For arbitrary design parameters $\gamma, \lambda > 0$, $r, s \geqslant 1$ and $\hat{u}(\cdot) \in L_\infty(\mathbb{R}_+, \mathbb{R}^m)$, we want that the adaptive feedback controller, consisting of the feedback law (12) together with the gain adaptation (13), whenever applied to a system (11) satisfying the assumptions (A1)–(A4) with arbitrary $k_0 \in \mathbb{R}, y_0 \in \mathbb{R}^m, z_0 \in \mathbb{R}^p, y_{\mathrm{ref}}(\cdot), n(\cdot) \in \mathcal{W}^{1,\infty}$, permits a solution (14) on a maximal time interval $[0, \omega)$, for some $\omega \in (0, \infty]$, and every maximal solution $(y(\cdot), z(\cdot), k(\cdot))$: $[0, \omega) \to \mathbb{R}^{m+p+1}$ is such that the following control objectives hold:

$$\left.\begin{array}{ll} \text{(i)} & \omega = \infty \\ \text{(ii)} & \lim_{t\to\infty} k(t) = k_\infty \quad \text{exists and is finite} \\ \text{(iii)} & z(\cdot) \in L_\infty(\mathbb{R}_+, \mathbb{R}^p) \\ \text{(iv)} & e(t) = y(t) - y_{\mathrm{ref}}(t) - n(t) \quad \text{approaches the ball } \mathcal{B}_\lambda(0) \quad \text{as} \quad t \to \infty \end{array}\right\} \qquad (15)$$

3 λ-tracking

We are now in a position to state the main result, i.e. if the system (11) satisfies the 'abstract' conditions (A1)–(A4), then the λ-tracker (12), (13) can be applied to ensure the control objectives (15). This result is fairly general. It clearly distinguishes between essential properties of the system and the adaptation mechanism. Classes of systems which actually satisfy these conditions are given in Sections 4–6.

Theorem 1. Suppose $\gamma, \lambda > 0$, $r, s \geqslant 1$, and $\hat{u}(\cdot) \in L_\infty(\mathbb{R}_+, \mathbb{R}^m)$. Let (11), (12) be such that (A1)–(A4) hold. If the gain adaptation (13) is applied to (11), (12), then, for all initial conditions (y_0, z_0, k_0), every maximal solution of (11)–(13) meets the control objectives (15).

For the proof of Theorem 1 we need the following technical lemma.

Lemma 1. Suppose $\xi(\cdot) \in \mathrm{L}_q(\mathbb{R}_+, \mathbb{R}^m)$ for some $q \geqslant 1$ and absolutely continuous on any compact interval. If there exist some $L, L' > 0$ such that, for almost all $t \geqslant 0$

$$\| \dot{\xi}(t) \| \leqslant L' \qquad \text{if} \qquad \| \xi(t) \| \leqslant L$$

then

$$\lim_{t\to\infty} \xi(\mathrm{t}) = 0$$

Proof. Consider the absolutely continuous function

$$\phi(\cdot) \colon \mathbb{R}_+ \to \mathbb{R}_+, \quad t \mapsto \min\{ \| \xi(t) \|^q, L^q \}$$

By the hypotheses of Lemma 1, $\phi(\cdot) \in L_1(\mathbb{R}_+, \mathbb{R}_+)$ with derivative $|\dot{\phi}(t)| \leqslant qL^{q-1}L'$ for almost all t. Therefore, $\phi(\cdot) \in L_1(\mathbb{R}_+, \mathbb{R}_+)$ is uniformly continuous and so, by Barbălat's Lemma (see Barbălat (1959) or Khalil (1996)), $\lim_{t\to\infty} \phi(t) = 0$. Now the claim of Lemma 1 follows and the proof is complete.

Proof of Theorem 1. It follows from the theory of ordinary differential equations that the closed-loop system (11)–(13) admits a solution (14) maximally extended over some interval $[0, \omega)$, $\omega \in (0, \infty]$.

STEP 1: We prove $k(\cdot) \in L_\infty([0, \omega), \mathbb{R})$.

Seeking a contradiction suppose $k(\cdot)$ is unbounded on $[0, \omega)$. Then the high-gain property (A1) yields that $e(t)$ tends to 0 as t tends to ω and hence, by the dead-zone incorporated in the gain adaptation, it follows that $k(\cdot)$ is bounded. This contradiction proves $k(\cdot) \in L_\infty([0, \omega), \mathbb{R})$.

STEP 2: We prove $\omega = \infty$.
Since by Step 1 and (A2) the components $k(\cdot)$ and $z(\cdot)$ of the solution are essentially bounded, and by (A3) the component $y(\cdot)$ does not have a finite escape time, it follows that the maximal interval of the solution is $[0, \infty)$.

STEP 3: We prove $\lim_{t\to\infty} d_\lambda(e(t)) = 0$.
On any compact interval we have: $e(\cdot)$ and $\|\cdot\|$ are absolutely continuous and the composition $t \mapsto \|e(t)\|$ is of bounded variation. Hence $t \mapsto \|e(t)\|$ is absolutely continuous; see, e.g. p. 297 of Hewitt and Stromberg (1965). An analogous argument proves absolute continuity of $t \to d_\lambda(e(t))$. Now a routine calculation gives, for almost all $t \geqslant 0$ with $e(t) \neq 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|e(t)\| = \frac{\langle e(t), \dot{e}(t)\rangle}{\|e(t)\|}$$

and hence, for almost all $t \geqslant 0$

$$\frac{\mathrm{d}}{\mathrm{d}t} d_\lambda(e(t)) \leqslant \|\dot{e}(t)\| \tag{16}$$

For almost all $t \geqslant 0$, we may conclude, from the boundedness of $y_{\mathrm{ref}}(\cdot) + n(\cdot)$, (A4), boundedness of $\dot{y}_{\mathrm{ref}}(\cdot) + \dot{n}(\cdot)$ and (16), that for $M > 0$ there exist $L, L', M' > 0$ so that

$$d_\lambda(e(t)) < M \;\Rightarrow\; \|e(t)\| < M + \lambda \;\Rightarrow\; \|y(t)\| < L \;\Rightarrow\; \|\dot{y}(t)\| < L'$$

$$\Rightarrow\; \|\dot{e}(t)\| < M' \qquad \Rightarrow\; \frac{\mathrm{d}}{\mathrm{d}t} d_\lambda(e(t)) < M'$$

Thus the presuppositions of Lemma 1 hold and hence $\lim_{t\to\infty} d_\lambda(e(t)) = 0$. This completes the proof.

4 Polynomial minimum phase systems in input affine form

In this section, we will show that multivariable polynomial systems in input affine form with strong relative degree-one and globally exponentially stable zero dynamics satisfies our assumptions (A1)–(A4) and hence the adaptive controller (12), (13) achieves λ-tracking.

We consider multivariable nonlinear systems in input affine form

$$\boxed{\begin{aligned} \dot{y}(t) &= f(t, y(t), z(t)) + G(t, y(t), z(t))\, u(t), & y(0) &= y_0 \\ \dot{z}(t) &= h(t, y(t), z(t)), & z(0) &= z_0 \end{aligned}} \tag{17}$$

where, for $m, p \in \mathbb{N}$, $y_0 \in \mathbb{R}^m$, $z_0 \in \mathbb{R}^p$ and

$$f\colon\ \mathbb{R}_+ \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^m$$

$$G\colon\ \mathbb{R}_+ \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^{m\times m}$$

$$h\colon\ \mathbb{R}_+ \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^p$$

are assumed to be Carathéodory functions.

Let $(y_e, z_e, u_e) \in \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^m$ denote an equilibrium point of (17), i.e. for all $t \in \mathbb{R}_+$

$$\begin{aligned} 0 &= f(t, y_e, z_e) + G(t, y_e, z_e) u_e \\ 0 &= h(t, y_e, z_e) \end{aligned} \tag{18}$$

Then we assume:

(NL1) The functions $f(\cdot)$ and $G(\cdot)$ are uniformly polynomially bounded, i.e. for some (unknown) polynomials $p_f(\cdot), p_G(\cdot) \in \mathbb{R}[s]$ we have, for all $(t, y, z) \in \mathbb{R}_+ \times \mathbb{R}^m \times \mathbb{R}^p$

$$\| f(t, y, z) \| \leqslant p_f\left(\left\| \begin{matrix} y \\ z \end{matrix} \right\| \right)$$

$$\| G(t, y, z) \| \leqslant p_G\left(\left\| \begin{matrix} y \\ z \end{matrix} \right\| \right)$$

An upper bound $s \geqslant 1$ of the degree of the polynomials is known:

$$s \geqslant \max\{\deg p_f(\cdot), \deg p_G(\cdot)\}$$

and, for some (unkown) $L_h > 0$ and all $(t, y, z) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^p$

$$\| h(t, y, z) - h(t, y_e, z) \| \leqslant L_h [1 + \| y - y_e \|]$$

(NL2) There exists some (unknown) positive-definite $P = P^T \in \mathbb{R}^{m \times m}$ such that, for all $(t, y, z) \in \mathbb{R}_+ \times \mathbb{R}^m \times \mathbb{R}^p$

$$2I_m \leqslant PG(t, y, z) + G(t, y, z)^T P$$

(NL3) The zero dynamics

$$\dot{\eta}(t) = h(t, y_e, \eta(t)), \qquad \eta(0) = \eta_0$$

are globally exponentially stable at z_e in the sense that there exists a positive-definite Lyapunov function

$$W(\cdot, \cdot) \colon [0, \infty) \times \mathbb{R}^p \to [0, \infty)$$

with constants $w_1, w_2, w_3, w_4 > 0$ and $q \geqslant 1$ so that, for all $(t, z) \in \mathbb{R}_+ \times \mathbb{R}^p$

$$w_1 \| z - z_e \|^q \leqslant W(t, z - z_e) \leqslant w_2 \| z - z_e \|^q \tag{19a}$$

$$\left\| \frac{\partial}{\partial z} W(t, z) \right\| \leqslant w_3 \| z \|^{q-1} \tag{19b}$$

$$\frac{\partial}{\partial t} W(t, z) + \left\langle \frac{\partial}{\partial z} W(t, z), h(t, y_e, z) \right\rangle \leqslant -w_4 \| z \|^q \tag{19c}$$

Remark 1. For linear systems the assumptions (NL1)–(NL3) are explained in Remark 3. For nonlinear systems they mean the following:

(i) Assumption (NL2) implies 'strong relative degree-one', see Section 5.1 of

Isidori (1995). If the system (17) is single-input, single-output, then (NL2) simplifies to requiring that $G(t, y, z)$ is uniformly bounded away from zero.

(ii) Assumption (NL3) requires the system to be globally minimum phase. This is, together with the assumption that the relative degree is one, a restriction. However, many practical control problems, like for example many chemical and biochemical reactors, will meet the requirements.

(iii) The exponent q in (19a)–(19c) can be replaced by any $\tilde{q} \geqslant 1$ and appropriate constants $\tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \tilde{w}_4 > 0$. To see this use the transformation $\tilde{W}(t, z) := W(t, z)^{\tilde{q}/q}$. Sufficient conditions for (19a)–(19c) are given in Hahn (1967) or Vidyasagar (1993).

Remark 2. The system (17) will be simplified by a straightforward coordinate transformation which converts the equilibrium (y_c, z_c) into zero:

$(y(t), z(t))$ is a solution of (17) if, and only if

$$(\tilde{y}(t), \tilde{z}(t)) := (y(t) - y_c, z(t) - z_c)$$

is a solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{y}(t) = \tilde{f}(t, \tilde{y}(t), \tilde{z}(t)) + \tilde{G}(t, \tilde{y}(t), \tilde{z}(t))\tilde{u}(u(t))$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{z}(t) = \tilde{h}(t, \tilde{z}(t)) \qquad + \bar{h}(t, \tilde{y}(t), \tilde{z}(t))$$

where

$$\tilde{f}(t, \tilde{y}, \tilde{z}) := f(t, \tilde{y} + y_c, \tilde{z} + z_c) - f(t, y_c, z_c) + [G(t, \tilde{y} + y_c, \tilde{z} + z_c) - G(t, y_c, z_c)]u_c$$

$$\tilde{G}(t, \tilde{y}, \tilde{z}) := G(t, \tilde{y} + y_c, \tilde{z} + z_c)$$

$$\tilde{u}(u) := u - u_c$$

$$\tilde{h}(t, \tilde{z}) := h(t, y_c, \tilde{z} + z_c)$$

$$\bar{h}(t, \tilde{y}, \tilde{z}) := h(t, \tilde{y} + y_c, \tilde{z} + z_c) - h(t, y_c, \tilde{z} + z_c)$$

This follows easily by rearranging (17) and using the fact that (y_c, z_c, u_c) is an equilibrium point. (A1), (A2) and (A4) yield that

$$\tilde{f}(t, 0, 0) = 0, \quad \tilde{G}(t, 0, 0) = G(t, y_c, z_c), \quad \tilde{h}(t, 0) = \bar{h}(t, 0, \tilde{z}) = 0, \quad \text{and} \quad \tilde{u}(u_c) = 0$$

whence $(0, 0, 0)$ is an equilibrium point.

Since

$$(a + b)^q \leqslant 2^{q-1}[|a|^q + |b|^q] \qquad \text{for any} \quad a, b \in \mathbb{R} \quad \text{and} \quad q \geqslant 1 \tag{20}$$

(see, e.g. Section XI.4 of Lang (1969), it easily follows that the polynomial bounds in (NL1) yield, for some $M > 0$ and for all $(t, \xi, \eta) \in \mathbb{R}_+ \times \mathbb{R}^m \times \mathbb{R}^p$

$$\begin{aligned} \|\tilde{f}(t, \xi, \eta)\| + \|\tilde{G}(t, \xi, \eta)\| &\leqslant M\left[1 + \left\|\begin{matrix}\xi \\ \eta\end{matrix}\right\|^s\right] \\ \|\bar{h}(t, \xi, \eta)\| &\leqslant M[+\|\xi\|] \end{aligned} \tag{21}$$

Furthermore, $\tilde{h}(\cdot)$ is a Carathéodory function, and the zero dynamics $\dot{\eta}(t) = \tilde{h}(t, \eta(t))$ are globally exponentially stable at 0, i.e. (19a)–(19c) hold true analogously.

Remark 3 (linear minimum phase systems). An important subclass (see Section 2.1 of Ilchmann (1993) for details) of systems (17) satisfying (NL1)–(NL3) is the class of multivariable linear systems which are minimum phase and the spectrum of their 'high-frequency gain' lies in $\mathbb{C}_+$, i.e.

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x(0) = x_0 \\ y(t) &= Cx(t) \end{aligned} \tag{22}$$

where $A \in \mathbb{R}^{n\times n}$, $B, C^{\mathrm{T}} \in \mathbb{R}^{n\times m}$, $x_0 \in \mathbb{R}^n$ satisfy:

(L1) $\sigma(CB) \subset \mathbb{C}_+$,

(L2) $\det\begin{bmatrix} sI_n - A & B \\ C & 0 \end{bmatrix} \neq 0$ for all $s \in \overline{\mathbb{C}_+}$.

First note, that, since $\det CB \neq 0$, a state space transformation according to $\mathbb{R}^n = \operatorname{im} B \oplus \ker C$, converts (22) into the form

$$\begin{aligned} \dot{y}(t) &= A_1 y(t) + A_2 z(t) + CB\, u(t) \\ \dot{z}(t) &= A_3 y(t) + A_4 z(t) \end{aligned} \tag{23}$$

(23) is exactly of the form (17) and we will compare the assumptions. Assumption (NL1) is trivially satisfied. By (L1) there exists a positive-definitive solution $P = P^{\mathrm{T}} \in \mathbb{R}^{m\times m}$ of the Lyapunov equation $2I_m = PCB + (CB)^{\mathrm{T}} P$ and hence (NL2) follows.

To verify (NL3), note that the zero dynamics of (22) respectively (23) are given by $\dot{z}(t) = A_4 z(t)$. It is easy to see that (L2) yields $\sigma(A_4) \subset \mathbb{C}_-$. Hence there exists a positive-definite solution $Q = Q_{\mathrm{T}} \in \mathbb{R}^{(n-m)\times(n-m)}$ of the Lyapunov equation $-I_{n-m} = QA_4 + A_4^{\mathrm{T}} Q$ and it is straightforward to check that $W(t, z) := z^{\mathrm{T}} Qz$ satisfies (NL3).

In the remainder of this section we will show that the class of nonlinear systems (17) satisfying (NL1)–(NL3) and with control (12) belong to the class (11)–(12) with (A1)–(A4). The most crucial property to show is (A1).

Lemma 2 (high-gain lemma for polynomial systems). Suppose $y_{\mathrm{ref}}(\cdot), n(\cdot) \in \mathcal{W}^{1,\infty}$, $\hat{u}(\cdot) \in L_\infty(\mathbb{R}_+, \mathbb{R}^m)$ and $k(\cdot)$: $[0, \omega) \to \mathbb{R}$ is a continuous, monotonically non-decreasing, unbounded function. Then $\lim_{t\to\omega} e(t) = 0$ for every maximal solution $(y(\cdot), z(\cdot), k(\cdot))$: $[0, \omega) \to \mathbb{R}^{m+p+1}$ of (17) with control (12).

Proof. The closed-loop system (17), (12) may be written as

$$\left.\begin{aligned} \dot{e}(t) &= f(t, e(t) + w(t), z(t)) - k(t)G(t, e(t) + w(t), z(t))\,\|e(t)\|^{s-1}e(t) \\ &\quad + G(t, e(t) + w(t), z(t))\hat{u}(t) - \dot{w}(t) \\ \dot{z}(t) &= h_0(t, z(t)) + \bar{h}(t, e(t) + w(t), z(t)) \end{aligned}\right\} \tag{24}$$

where

$$\left.\begin{aligned} w(t) &:= y_{\mathrm{ref}}(t) + n(t) \\ h_0(t, z) &:= h(t, y_e, z(t)) \\ \bar{h}(t, y, z) &:= \bar{h}(t, y, z) - h(t, y_e, z) \end{aligned}\right\} \tag{25}$$

In the following we will assume, by Remarks 2 and 1(iii) and without restriction of generality, that the origin is zero and (21) are satisifed.

For arbitrary $\rho > 0$ and positive-definite P as given in (NL2), we will prove the statement of Lemma 2 by showing that the Lyapunov-like candidate (see Nomenclature)

$$V_\rho(e(t)) := \frac{1}{2} D_\rho(e(t))^2$$

tends to 0 as t tends to ω. This would complete the proof since $\rho > 0$ is arbitrary. Before we proceed in several steps, we record, also for later use, the following facts:

$$\text{If } p_1 := \|P^{-1}\|^{-1/2}, p_2 := \|P\|^{1/2}, \text{ then } p_1\|e\| \leqslant \|e\|_P \leqslant p_2\|e\| \text{ for all } e \in \mathbb{R}^m \quad (26)$$

$$\text{If } D_\rho(e) > 0, \text{ then } 1 < \frac{1}{\rho}\|e\|_P < \frac{p_2}{\rho}\|e\| \quad (27)$$

STEP 1: We prove that differentiation of the C^1-function $V_\rho(e(t))$ along the solution component $e(t)$ of (24) yields, for suitable $M_1 > 0$, $t_1 \in (0, \omega)$ and almost all $t \in [t_1, \omega)$

$$\frac{\mathrm{d}}{\mathrm{d}t} V_\rho(e(t)) \leqslant -\tilde{k}(t) D_\rho(e(t))\|e(t)\|^s + M_1\|\Theta_\rho(e(t))\|\,\|z(t)\|^s \quad (28)$$

where $\tilde{k}(\cdot)$ has the same properties as $k(\cdot)$, i.e. non-decreasing and unbounded on $[0, \omega)$. Differentiation along (24) and applying (NL2) yields, for almost all $t \in [0, \omega)$, (where for the sake of presentation we omit the argument t)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} V_\rho(e(t)) &= \frac{D_\rho(e(t))}{\|e(t)\|_P}\langle e(t), P\dot{e}(t)\rangle \\ &\leqslant -k\sigma\frac{D_\rho(e)}{\|e\|_P}\|e\|^{s+1} + \frac{D_\rho(e)}{\|e\|_P}\|e\| \cdot \|P\| \cdot \|f(e+w, z) + G(e+w, z)\hat{u} - \dot{w}\|\end{aligned}$$

Using the bounds in (21) and (20), boundedness of $w(\cdot)$ and $\dot{w}(\cdot)$, and $\Theta_\rho(e)$ as defined in the Nomenclature, it is readily verified that there exists some $M_2 > 0$ such that, for almost all $t \in [0, \omega)$

$$\frac{\mathrm{d}}{\mathrm{d}t} V_\rho(e(t)) \leqslant -k\sigma\frac{D_\rho(e)}{\|e\|_P}\|e\|^{s+1} + M_2\|\Theta_\rho(e)\|[1 + \|z\|^s + \|e\|^s]$$

Choosing $t_1 \in (0, \omega)$ sufficiently large so that $k(t_1) > 0$ and invoking (26), (27) yields, for almost all $t \in [t_1, \omega)$

$$\frac{\mathrm{d}}{\mathrm{d}t} V_\rho(e(t)) \leqslant -k\frac{\sigma}{p_2} D_\rho(e)\|e\|^s + M_2 D_\rho(e)\frac{1}{p_1}[1 + \|e\|^s] + M_2\|\Theta_\rho(e)\|\,\|z\|^s$$

Now (28) follows by a repeated application of (27) and by setting

$$\tilde{k}(t) := k(t)\frac{\sigma}{p_2} - \frac{M_2}{p_1}\left[\left(\frac{p_2}{\rho}\right)^s + 1\right]$$

STEP 2: We prove that the solution component $z(t)$ of (24) satisfies, for suitable $M_3 > 0$ and all $t \in [0, \omega)$

$$\| z(t) \|^s \leqslant M_3 + M_3 \mathscr{L}(\| \Theta_\rho(e(\cdot)) \|)(t)^s \tag{29}$$

where, for some $\mu > 0$, $\mathscr{L}$ is the convolution operator

$$\mathscr{L}: L_q([0,\omega), \mathbb{R}_+) \to L_q([0,\omega), \mathbb{R}_+), \qquad \varphi(\cdot) \mapsto \left(t \mapsto \int_0^t e^{-\mu(t-\tau)} \varphi(\tau)\, d\tau \right) \tag{30}$$

Note that $\mathscr{L}$ is a linear and bounded operator for each $q \in [1, \infty]$, see e.g. Section 6.4 of Vidyasagar (1993).

In order to derive (29), we differentiate the Lyapunov function in (NL3) along the solution component z(t) in (24) and obtain, for almost all $t \in [0, \omega)$

$$\frac{d}{dt} W(t, z(t)) = \frac{\partial}{\partial t} W(t, z(t)) + \left\langle \frac{\partial}{\partial z} W(t, z(t)), h_0(t, z(t)) + \bar{h}(t, e(t) + w(t), z(t)) \right\rangle$$

Invoking the bounds in (NL3) and in (NL1) respectively (21) we obtain, for $q = 1$ and some $M_4 > 0$

$$\begin{aligned} \frac{d}{dt} W(t, z(t)) &\leqslant -w_4 \| z(t) \| + w_3 M_4 [1 + \| e(t) + w(t) \|] \\ &\leqslant -\frac{w_4}{w_1} W(t, z(t)) + w_3 M_4 [1 + \| e(t) + w(t) - \Theta_\rho(e(t)) \| + \| \Theta_\rho(e(t)) \|] \end{aligned}$$

and hence, with

$$\mu := \frac{w_4}{w_1}, \qquad M_5 := w_3 M_4 + \| e(\cdot) + w(\cdot) - \Theta_\rho(e(\cdot)) \|_{L_\infty(0,\omega)}$$

we derive at

$$\frac{d}{dt} W(t, z(t)) \leqslant -\mu W(t, z(t)) + M_5 [1 + \| \Theta_\rho(e(t)) \|] \tag{31}$$

which is equivalent to

$$\frac{d}{dt} [e^{\mu t} W(t, z(t))] \leqslant e^{\mu t} M_5 [1 + \| \Theta_\rho(e(t)) \|]$$

and therefore, for almost all $t \in [0, \omega)$

$$W(t, z(t)) \leqslant e^{-\mu t} W(t, z_0) + M_5 \int_0^t e^{-\mu(t-\tau)} [1 + \| \Theta_\rho(e(\tau)) \|] d\tau \tag{32}$$

Applying (NL3) to (32) yields

$$\| z(t) \| \leqslant \frac{w_2}{w_1} e^{-\mu t} \| z_0 \| + \frac{M_5}{w_1 \mu} + \frac{M_5}{w_1} \mathscr{L}(\| \Theta_\rho(e(\cdot)) \|)(t)$$

and by using (20) we may conclude (29).

STEP 3: We prove that, for suitable $M_6 > 0$ and for all $t \in [0, \omega)$, we have

$$\int_0^t \| \Theta_\rho(e(\tau)) \| \, \| z(\tau) \|^s \, \mathrm{d}\tau \leqslant M_6 [\| \Theta_\rho(e(\cdot)) \|_{L_1(0,t)} + \| \Theta_\rho(e(\cdot)) \|^{s+1}_{L_{s+1}(0,t)}] \tag{33}$$

Integration of (29) gives, for all $t \in [0, \omega)$

$$\int_0^t \| \Theta_\rho(e(\tau)) \| \, \| z(\tau) \|^s \, \mathrm{d}\tau \leqslant M_3 \int_0^t \| \Theta_\rho(e(\tau)) \| \, \mathrm{d}\tau + M_3 \int_0^t \| \Theta_\rho(e(\tau)) \| \, \| \mathscr{L}(\| \Theta_\rho(e) \|)(\tau) \|^s \, \mathrm{d}\tau$$

and thus, by Hölder's inequality for $1/p + 1/q = 1$ and uniform boundedness of the operator $\mathscr{L}$

$$\begin{aligned}
\int_0^t \| \Theta_\rho(e(\tau)) \| \, \| z(\tau) \|^s \, \mathrm{d}\tau &\leqslant M_3 \int_0^t \| \Theta_\rho(e(\tau)) \| \, \mathrm{d}\tau \\
&\quad + M_3 \int_0^t \| \Theta_\rho(e(\tau)) \| \, \mathscr{L}(\| \Theta_\rho(e(\tau)) \|)^s \, \mathrm{d}\tau \\
&\leqslant M_3 \| \Theta_\rho(e(\cdot)) \|_{L_1(0,t)} \\
&\quad + M_3 \| \Theta_\rho(e(\cdot)) \|_{L_p(0,t)} \| \mathscr{L}(\Theta_\rho(e))(\cdot) \|^s_{L_{sq}(0,t)} \\
&\leqslant M_3 \| \Theta_\rho(e(\cdot)) \|_{L_1(0,t)} \\
&\quad + M_3 \| \mathscr{L} \|^s \| \Theta_\rho(e(\cdot)) \|_{L_p(0,t)} \| \Theta_\rho(e(\cdot)) \|^s_{L_{sq}(0,t)}
\end{aligned}$$

Setting $p = s + 1$, $q = (s + 1)/s$ and $M_6 = M_3[1 + \| \mathscr{L} \|^s]$ yields (33).

STEP 4: We prove that $V_\rho(e(\cdot)) \in L_\infty([0, \omega), \mathbb{R}_+)$.
Integration of (28) and inserting (33) yields, for all $t \in [t_1, \omega)$

$$\begin{aligned}
V_\rho(e(t)) &\leqslant V_\rho(e(t_1)) - \int_{t_1}^t \tilde{k}(\tau) D_\rho(e(\tau)) \| e(\tau) \|^s \, \mathrm{d}\tau \\
&\quad + M_1 \int_{t_1}^t \| \Theta_\rho(e(\tau)) \| \, \| z(s) \|^s \, \mathrm{d}\tau \\
&\leqslant V_\rho(e(t_1)) - \int_{t_1}^t \tilde{k}(\tau) D_\rho(e(\tau)) \| e(\tau) \|^s \, \mathrm{d}\tau \\
&\quad + M_1 M_6 \left[\int_{t_1}^t \| \Theta_\rho(e(\tau)) \| + \| \Theta_\rho(e(\tau)) \|^{s+1} \, \mathrm{d}\tau \right]
\end{aligned}$$

Choosing $t_2 \in (0, \omega)$ sufficiently large so that $\tilde{k}(t_2) > 0$ and setting

$$M_7 := V_\rho(e(t_1)) - \int_{t_1}^{t_2} \tilde{k}(\tau) D_\rho(e(\tau)) \| e(\tau) \|^s \mathrm{d}\tau$$

$$+ M_1 M_6 \int_{t_1}^{t_2} \left[\| \| \Theta_\rho(e(\tau)) \|^s + \| \Theta_\rho(e(\tau)) \|^{s+1} \right] \mathrm{d}\tau$$

gives, for all $t \in [t_2, \omega)$

$$V_\rho(e(t)) \leqslant M_7 - \int_{t_2}^{t} \tilde{k}(\tau) D_\rho(e(\tau)) \| e(\tau) \|^s \mathrm{d}\tau + M_1 M_6 \int_{t_2}^{t} \left[\| \Theta_\rho(e(\tau)) \| + \| \Theta_\rho(e(\tau)) \|^{s+1} \right] \mathrm{d}\tau \tag{34}$$

It is easily derived from (26) and (27) that

$$\| \Theta_\rho(e) \| \leqslant \frac{1}{p_1} D_\rho(e) \leqslant \frac{1}{p_1} D_\rho(e) \left(\frac{p_2}{\rho} \right)^s \| e \|^s$$

and that, if $D_\rho(e) > 0$

$$- \| e \|^s \leqslant - \frac{1}{p_2^s} D_\rho(e)^s \qquad \text{and} \qquad \| \Theta_\rho(e) \|^{s+1} \leqslant \frac{1}{p_1^{s+1}} D_\rho(e)^{s+1}$$

These inequalities applied to (34) yield, for all $t \in [t_2, \omega)$

$$\begin{aligned} V_\rho(e(t)) \leqslant M_7 &- \int_{t_0}^{t} \left[\frac{1}{2} \tilde{k}(\tau) - \frac{M_1 M_6}{p_1} \left(\frac{1}{\rho} \right)^s \right] D_\rho(e(\tau)) \| e(\tau) \|^s \mathrm{d}\tau \\ &- \int_{t_0}^{t} \left[\frac{1}{2} \tilde{k}(\tau) \frac{1}{p_2^s} - \frac{M_1 M_6}{p_1^{s+1}} \right] D_\rho(e(\tau))^{s+1} \mathrm{d}\tau \end{aligned} \tag{35}$$

Since $k(t)$ is assumed to be unbounded on $[0, \omega)$, $\tilde{k}(t)$ is, and boundedness of $V_\rho(e(\cdot))$ follows from (35).

STEP 5: We prove that $\lim_{t \to \omega} D_\rho(e(t)) = 0$.
Since $V_\rho(e(\cdot))$ is bounded, it follows that $e(\cdot) \in \mathrm{L}_\infty([0, \omega), \mathbb{R}^m)$ and hence, by (29) and boundedness of $\mathscr{L}: L_\infty([0, \omega), \mathbb{R}_+) \to L_\infty([0, \omega), \mathbb{R}_+)$, we derive $z(\cdot) \in L_\infty([0, \omega), \mathbb{R}^m)$. Applying (27) to (28) yields, for almost all $t \in [t_1, \omega)$

$$\frac{\mathrm{d}}{\mathrm{d}t} V_\rho(e(t)) \leqslant - D_\rho(e(\tau)) \left[\tilde{k}(t) \left(\frac{\rho}{p_2} \right)^{s-1} - \frac{M_1}{\rho} \| z(t) \|^s \right] \| e(t) \| \tag{36}$$

Since $z(\cdot) \in L_\infty([0,\omega), \mathbb{R}^m)$ we may choose $t_3 \in [t_1, \omega)$ such that

$$\hat{k}(t) := \tilde{k}(t)\left(\frac{\rho}{p_2}\right)^{s-1} - \frac{M_1}{\rho}\|z(t)\|^s > 0 \qquad \text{for all } t \in [t_3, \omega)$$

and hence, by (36), for almost all $t \in [t_3, \omega)$

$$\frac{\mathrm{d}}{\mathrm{d}t} V_\rho(e(t)) \leqslant -\hat{k}(t) D_\rho(e(t)) \|e(t)\| \leqslant -2\hat{k}(t) V_\rho(e(t))$$

Since $\hat{k}(\cdot)$ is unbounded on $[0, \omega)$, exponential decay of $V_\rho(e(t))$ on $[0, \omega)$ to 0 follows. This completes the proof.

We are now in a position to complete this section by proving that the assumptions (NL1)–(NL3) are sufficient for (A1)–(A4).

Proposition 1. Suppose $y_{\mathrm{ref}}(\cdot), n(\cdot) \in \mathcal{W}^{1,\infty}$ and $\tilde{u}(\cdot) \in L_\infty(\mathbb{R}_+, \mathbb{R}^m)$. Then the class of nonlinear minimum phase systems (17) satisfying (NL1)–(NL3) and with control (12) is a subset of the class of systems (11), (12) satisfying (A1)–(A4). Hence the control objectives (15) are met if (12), (13) are applied to (17) with s as given in (NL1) and arbitrary $\gamma, \lambda > 0, r \geqslant s, k_0 \in \mathbb{R}$.

Proof. Substituting the feedback (12) into (17) yields, for $(\tilde{y}(t), \tilde{z}(t)) := (y(t) - y_c, z(t) - z_c)$, the initial-value problem

$$\left.\begin{aligned} \dot{\tilde{y}}(t) &= f(t, \tilde{y}(t) + y_c, \tilde{z}(t) + z_c) - k(t) G(t, \tilde{y}(t) + y_c, z(t) + z_c) \|e(t)\|^{s-1} e(t) \\ &\qquad + G(t, \tilde{y}(t) + y_c, z(t) + z_c)\tilde{u}(t) \\ \dot{\tilde{z}}(t) &= h_0(t, \tilde{z}(t) + z_c) \qquad + \bar{h}(t, \tilde{y}(t) + y_c, \tilde{z}(t) + z_c) \end{aligned}\right\} \qquad (37)$$

where $\tilde{y}(0) = y_0 - y_c, \tilde{z}(0) = z_0 - z_c$ and $h_0(\cdot), \bar{h}(\cdot)$ are as in (23).

By the classical theory of ordinary differential equations, the initial-value problem (37), (13) has a solution, maximally extended over $[0, \omega)$, for some $\omega \in (0, \infty]$. In order to apply Theorem 1, it remains to prove properties (A1)–(A4).

(A1): This follows from Lemma 2.
(A2): Suppose $k(\cdot) \in L_\infty([0.\omega), \mathbb{R})$. To prove boundedness of $z(\cdot)$, we first observe (see Lemma 2 in Ilchmann and Ryan (1994)), that for $\rho = p_2\lambda$

$$\|\theta_\rho(e)\| \leqslant \frac{\|e\|}{\|e\|_P} D_\rho(e) \leqslant \frac{1}{p_1} D_\rho(e) \leqslant \frac{p_2}{p_1} d_\lambda(e)$$

Substituting this into (29) yields, for almost all $t \in [0, \omega)$

$$\|z(t)\|^s \leqslant M_3 + M_3\left(\frac{p_2}{p_1}\right)^s \mathscr{L}(d_\lambda(e(\cdot)))(t)^s$$

Now the statement follows from the fact that $k(\cdot) \in L_\infty([0, \omega), \mathbb{R})$ is equivalent to $d_\lambda(e) \in L_r([0, \omega), \mathbb{R})$ and $\mathscr{L}$ is a uniformly bounded operator.
(A3): Suppose $\omega < \infty$. It suffices to prove $e(\cdot) \in L_\infty([0, \omega), \mathbb{R}^m)$. Since $k(\cdot)$ and $z(\cdot)$

are bounded on the maximal interval of existence of a solution of (37), (13), we have

$$\lim_{t\to\omega} \| e(t) \| = \infty \tag{38}$$

Obviously, $t \mapsto e(t)$ and $e \mapsto \| e \|$ are absolutely continuous functions on any compact interval and the composition $t \mapsto \| e(t) \|$ is of bounded variation. Hence, it follows (see, e.g. p. 297 of Hewitt and Stromberg (1965)) that $t \mapsto \| e(t) \|$ is absolutely continuous and thus differentiable almost everywhere. By (38) we may choose $t' \in (0, \omega)$ such that $e(t) \neq 0$ for all $t \in [t', \omega)$. Then by (NL1), (NL2), (20), (26) and boundedness of $k(\cdot)$ and $z(\cdot)$, there exist M_1, M_2, $M_3 > 0$ such that, for almost all $t \in [t', \omega)$ with $e(t) \neq 0$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \| e(t) \|_P &= \| e(t) \|_P^{-1} \langle e(t), P\dot{e}(t) \rangle \\
&\leqslant -\frac{k(t)\sigma}{\| e(t) \|_P} \| e(t) \|^{s+1} - \| e(t) \|_P + \| e(t) \|_P + M_1 [1 + \| e(t) \|^s] \\
&\leqslant - \| e(t) \|_P + M_2 [1 + \| e(t) \|^s] \\
&\leqslant - \| e(t) \|_P + M_2 [1 + (d_\lambda(e(t)) + \lambda)^s] \\
&\leqslant - \| e(t) \|_P + M_2 [1 + (2\lambda)^s] + M_2 \frac{2^s}{\gamma} [1 + \dot{k}(t)] \\
&\leqslant - \| e(t) \|_P + M_3 [1 + \dot{k}(t)]
\end{aligned}
$$

Since $t \mapsto \| e(t) \|_P$ is absolutely continuous, the set

$$\mathcal{J}_1 := \{ t \in [0, \omega) \mid t \mapsto \| e(t) \|_P \text{ is not differentiable at } t \}$$

has measure zero, and since $t \mapsto \| e(t) \|_P$ is not differentiable at any point of

$$\mathcal{J}_2 := \{ t \in [0, \omega) \mid e(t) = 0, \dot{e}(t) \neq 0 \}$$

this set has measure zero, too. Integration of

$$\frac{\mathrm{d}}{\mathrm{d}t} \| e(t) \|_P = \begin{cases} \| e(t) \|_P^{-1} \langle e(t), P\dot{e}(t) \rangle, & t \in [0, \omega) \backslash \mathcal{J}_1 \cup \mathcal{J}_2 \quad \text{and} \quad e(t) \neq 0 \\ 0, & t \in [0, \omega) \backslash \mathcal{J}_1 \cup \mathcal{J}_2 \quad \text{and} \quad e(t) = 0 \end{cases}$$

and invoking the above inequality gives, by Lemma 3.2.4 in Ioannou and Sun (1996), for all $t \in [t', \omega)$

$$\| e(t) \|_P \leqslant \mathrm{e}^{-(t-t')} \| e(t') \|_P + \int_{t'}^{t} \mathrm{e}^{-(t-\tau)} M_3 [1 + \dot{k}(\tau)] \, \mathrm{d}\tau$$

Now boundedness of $k(\cdot)$ and $\omega < \infty$ yield boundedness of $\| e(\cdot) \|_P$. This contradicts (38) and proves the claim.

(A4): $k(\cdot) \in L_\infty(\mathbb{R}_+, \mathbb{R})$ is a consequence of (A1) and of the dead-zone incorporated in (13). Since $k(\cdot), z(\cdot), y_{\mathrm{ref}}(\cdot)$ are all bounded, the supposition in (A4) can be shown by invoking the bounds in (NL1). This completes the proof.

5 Polynomial minimum phase systems with sector bounded input nonlinearities

In this section, we consider multivariable nonlinear systems of the form

$$\boxed{\begin{aligned} \dot{y}(t) &= f(t, y(t), z(t)) + \chi(t, u(t)), \qquad & y(0) = y_0 \\ \dot{z}(t) &= h(t, y(t), z(t)), & z(0) = z_0 \end{aligned}} \tag{39}$$

where f and h are given as in (17), satisfy (NL1) and (NL3), and

$$\chi: \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}^m$$

is a Carathéodory function representing a 'sector bounded, memoryless, input nonlinearity' (see p. 403 of Khalil (1996)), i.e. for all $(t, u) \in \mathbb{R}_+ \times \mathbb{R}^m$

$$[\chi(t, u) - K_{\min} u]^{\mathrm{T}} [\chi(t, u) - K_{\max} u] \leqslant 0 \tag{40}$$

where, for some $0 < \alpha_i < \beta_i, i = 1, \ldots, m$

$$K_{\min} := \operatorname{diag}\{\alpha_1, \ldots, \alpha_m\}, \qquad K_{\max} := \operatorname{diag}\{\beta_1, \ldots, \beta_m\}$$

In the following we will use the fact that (40) is equivalent to

$$-u^{\mathrm{T}}[K_{\min} + K_{\max}]\chi(t, u) \leqslant -\|\chi(t, u)\|^2 - u^{\mathrm{T}} K_{\min} K_{\max} u \tag{41}$$

whence, for all $(t, u) \in \mathbb{R}_+ \times \mathbb{R}^m$

$$\|\chi(t, u)\| \leqslant \|K_{\min} + K_{\max}\| \, \|u\| \tag{42}$$

Lemma 3 (high-gain lemma for systems with sector bounded input nonlinearities). Suppose $y_{\text{ref}}(\cdot), n(\cdot) \in \mathcal{W}^{1,\infty}$, $\hat{u}(\cdot) \in L_\infty(\mathbb{R}_+, \mathbb{R}^m)$ is locally integrable and $k(\cdot)$: $[0, \omega) \to \mathbb{R}$ is a continuous, monotonically non-decreasing, unbounded function for some $\omega \in (0, \infty]$. Then $\lim_{t \to \omega} e(t) = 0$ for every maximal solution $(y(\cdot), z(\cdot), k(\cdot))$: $[0, \omega) \to \mathbb{R}^{m+p+1}$ of (39) with control (12).

Proof. The closed-loop system (12), (39) may be written as

$$\begin{aligned} \dot{e}(t) &= f(t, e(t) + w(t), z(t)) + \chi(t, -k(t)\|e(t)\|^{s-1} e(t) + \hat{u}(t)) - \dot{w}(t) \\ \dot{z}(t) &= h_0(t, z(t)) \qquad\qquad + \tilde{h}(t, e(t) + w(t), z(t)) \end{aligned} \tag{43}$$

where we use the notation as in (25) and assume again that $(y_c, z_c) = (0, 0)$ and $q = 1$ in (NL3).

Setting $P := K_{\min} + K_{\max}$ and differentiating $V_p(e(t))$ along (43) yields, for almost all $t \in [0, \omega)$ (where again for the sake of simplicity we omit the argument t)

$$\frac{\mathrm{d}}{\mathrm{d}t} V_p(e(t)) \leqslant \frac{D_p(e)}{\|e\|_P} \langle e, P\chi(-k\|e\|^{s-1} e + \hat{u}) + P[f(e + w, z) - \dot{w}] \rangle \tag{44}$$

For further bounds we proceed in several steps.

STEP 1: We prove that for $D_p(e) > 0$ and $k > 0$ we have, for suitable $M_1 < 0$

$$\langle e, P\chi(-k\|e\|^{s-1} e + \hat{u}) \rangle \leqslant -k\sigma_{\min}(\hat{K})\|e\|^{s+1} + M_1 \left[\frac{1}{k} + \|e\| \right] \tag{45}$$

Applying (41), (42), (26) and invoking boundedness of $\hat{u}(\cdot)$ yields, for some $M_2 > 0$ and $\hat{K} := K_{\min}K_{\max}$

$$
\begin{aligned}
\langle e, P\chi(-k\|e\|^{s-1}e+\hat{u})\rangle &= \frac{-1}{k\|e\|^{s-1}}\langle -k\|e\|^{s-1}e+\hat{u}, P\chi(-k\|e\|^{s-1}e+\hat{u})\rangle \\
&\quad + \frac{1}{k\|e\|^{s-1}}\langle \hat{u}, P\chi(-k\|e\|^{s-1}e+\hat{u})\rangle \\
&\leqslant -\frac{1}{k\|e\|^{s-1}}[\|\chi(-k\|e\|^{s-1}e+\hat{u})\|^2 \\
&\quad + [-k\|e\|^{s-1}e+\hat{u}]^{\mathrm{T}}\hat{K}[-k\|e\|^{s-1}e+\hat{u}]] \\
&\quad + \frac{1}{k\|e\|^{s-1}}\|\hat{u}\|\,\|P\|^2\|-k\|e\|^{s-1}e+\hat{u}\| \\
&\leqslant -\frac{1}{k\|e\|^{s-1}}[k^2\|e\|^{2(s-1)}\|e\|_{\hat{K}}^2 - 2k\|e\|^{s-1}e^{\mathrm{T}}\hat{K}\hat{u} + \|\hat{u}\|_{\hat{K}}^2] \\
&\quad + \frac{1}{k\|e\|^{s-1}}\|\hat{u}\|\,\|P\|^2[k\|e\|^s + \|\hat{u}\|] \\
&\leqslant -k\|e\|^{s-1}\|e\|_{\hat{K}}^2 + 2k\|e\|\,\|\hat{K}\|\,\|\hat{u}\| \\
&\quad + \|\hat{u}\|\,\|P\|^2\|e\| + \frac{\|P\|^2\|\hat{u}\|^2}{k\|e\|^{s-1}} \\
&\leqslant -k\sigma_{\min}(\hat{K})\|e\|^{s+1} + M_2\left[\|e\| + \frac{1}{k\|e\|^{s-1}}\right]
\end{aligned}
$$

An application of (27) shows (45).

STEP 2: We prove that for $t_1 \in (0, \omega)$ sufficiently large so that $\tilde{k}(t) > 0$, there exists some $M_3 > 0$ such that, for almost all $t \in [t_1, \omega)$

$$
\frac{\mathrm{d}}{\mathrm{d}t}V_\rho(e(t)) \leqslant -M_3[k(t)-1]D_\rho(e(t))\|e(t)\|^s + M_3\|\theta_\rho(e(t))\|\,\|z(t)\|^s \tag{46}
$$

Substituting (45) into (44), invoking the polynomial bound of f in (NL1), and applying (20), (27) and $w(\cdot), \dot{w}(\cdot) \in L_\infty(\mathbb{R}_+, \mathbb{R}^m)$ yields for some $M_4, M_5, M_6 > 0$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V_\rho(e(t)) &\leqslant -k\sigma_{\min}(\hat{K})\frac{D_\rho(e)}{\|e\|_P}\|e\|^{s+1} + M_1\frac{D_\rho(e)}{\|e\|_P}\left[\frac{1}{k(t_1)} + \|e\|\right] \\
&\quad + \frac{D_\rho(e)}{\|e\|_P}\|e\|\,\|P\|M_4\left[1 + \left\|\begin{matrix} e+w \\ z \end{matrix}\right\|^s + \|\dot{w}\|\right] \\
&\leqslant -\frac{D_\rho(e)}{\|e(t)\|_P}[M_4k\|e\|^{s+1} - M_4 - M_4\|e\| - M_4\|e\|2^{s/2}(\|e+w\|^s + \|e\|^s)]
\end{aligned}
$$

$$\leqslant -\frac{D_\rho(e)}{\|e(t)\|_P} M_4 \|e\|^{s+1}[k - \rho^{-(s+1)} - \rho^{-s} - M_5] + M_5 2^{s/2} \|\theta_\rho(e)\| \|z\|^s$$

$$\leqslant -\frac{D_\rho(e)}{\|e\|_P} M_4 \frac{\|e\|}{\|e\|_P} \|e\|^{s+1}[k - M_6] + M_6 \|\theta_\rho(e)\| \|z\|^s$$

If you choose $t_1 \in (0, \omega)$ sufficiently large so that $k(t_1) > M_6$, then an application of (26) to the above inequality yields (46).

STEP 3: The inequality (46) coincides, modulo other values for $k(\cdot)$ but same properties, with (28) and the remainder of the proof can be performed analogously to the proof of Lemma 2. This is omitted for brevity and the proof is complete.

We are now in a position to complete this section by proving that the class of systems (39) satisfying (NL1), (NL3) and the sector nonlinearity (40) can be regulated by the adaptive controller (11), (13) in the sense that the closed-loop system meets the control objectives (15).

Proposition 2. Suppose $y_{\mathrm{ref}}(\cdot), n(\cdot) \in \mathcal{W}^{1,\infty}$ and $\hat{u}(\cdot) \in \mathrm{L}_\infty(\mathbb{R}_+, \mathbb{R}^m)$. Then the class of nonlinear minimum phase systems (39) satisfying (NL1), (NL3), (40) and with control (12) is a subset of the class of systems (11), (12) satisfying (A1)–(A4). Hence the control objectives (15) are met if (12), (13) are applied to (39) with s as given in (NL1) and arbitrary $\gamma, \lambda > 0, r \geqslant s, k_0 \in \mathbb{R}$.

Proof. The closed-loop system may be written as (43) plus (13). Then it remains to prove that the properties (A1)–(A4) hold so that Theorem 1 can be applied. (A1) follows from Lemma 3. The proof of (A2) can be performed as in Proposition 1. To show (A3) and (A4), note that an application of (NL1) and (42) to the first equation in (39) yields

$$\|\dot{y}\| \leqslant M_f\left[1 + \left\|\begin{matrix} e \\ z \end{matrix}\right\|^s\right] + \|\chi(t, -ke\|e\|^{s-1} + \hat{u})\|$$

$$\leqslant M_f\left[1 + \left\|\begin{matrix} e \\ z \end{matrix}\right\|^s\right] + \|K_{\min} + K_{\max}\| \cdot \| - k\|e\|^{s-1}e + \hat{u}\|$$

Since $z(\cdot)$ and $k(\cdot)$ are bounded by (A1), (A2), similarly as in the proof of Proposition 1 one may show that (A3) and (A4) hold true. This completes the proof.

6 Systems with bounded trajectories

If systems (11) are not in input affine form (17) or (39), then it might be difficult or only locally possible to convert them into this form. However, there is a class of practical relevant systems where λ-tracking is possible without converting them into input affine form. Also the global minimum phase assumption need not be checked. The reason is, that under the feedback (12), the trajectories of certain biochemical processes remain bounded. Consider for example the general reactor model

$$\boxed{\dot{\xi}(t) = K\varphi(\xi(t)) - D(t)\xi(t) - \mathrm{diag}\{q_1, \ldots, q_N\}\xi(t) + D(t)\xi^{\mathrm{in}}(t), \quad \xi(0) \in (\mathbb{R}_+^\star)^N} \qquad (47)$$

where $\xi(t) \in \mathbb{R}^N$ is the state vector consisting of the concentrations, the N reaction rates $\varphi(\cdot) = (\varphi_1(\cdot), \dots, \varphi_n(\cdot))^{\mathrm{T}}$ are of the form

$$\varphi_j(\cdot)\colon \mathbb{R}_+^N \to \mathbb{R}_+, \qquad \xi \mapsto \varphi_j(\xi) = \alpha_j(\xi) \cdot \left(\prod_{k \in S_j} \xi_k \right)$$

where S_j is the set of autocatalysts and reactants of the jth reaction, the growth rates are given by

$$\alpha_j(\cdot)\colon \mathbb{R}_+^N \to (\underline{\alpha}_j, \bar{\alpha}_j], \qquad 0 \leqslant \underline{\alpha}_j \leqslant \bar{\alpha}_j, \qquad \text{for all} \qquad j = 1, \dots, M$$

$D(\cdot)$ denotes the dilution rate and the components $\xi_i^{\text{in}}(\cdot)$ of the inflow rate are bounded functions satisfying

$$\xi_i^{\text{in}}(\cdot)\colon \mathbb{R}_+ \to [0, \bar{\xi}_i^{\text{in}}], \qquad \bar{\xi}_i^{\text{in}} \geqslant 0, \qquad \text{for all} \qquad i = 1, \dots, N$$

the values of q_i are the non-negative proportional factors of the gaseous outflow rates, $K \in \mathbb{R}^{N \times M}$ is the matrix of stoichiometric coefficients. Variations of this model were studied by many authors and introduced in a unifying way by Bastin & Dochain (1990).

In Ilchmann and Weirig (1998) we proved the following.

Proposition 3. If for the stoichiometric matrix $K = [K_1, \dots, K_M]$ in (47) there

$$\text{exists some} \quad \gamma \in (\mathbb{R}_+^\star)^N \quad \text{such that} \quad \gamma^{\mathrm{T}} K_j \leqslant 0 \quad \text{for all} \quad j = 1, \dots, M \tag{48}$$

then for each initial condition $\xi(0) \in (\mathbb{R}_+^\star)^N$ the solution of (47) is bounded and remains in the positive orthant.

If K represents a 'non-cyclic process'—a concept introduced and characterized in Ilchmann and Weirig (1998)—then the crucial condition (48) is satisfied. Another sufficient condition is the assumption of mass conservation, see Gavalas (1968).

In the present paper we do not have the space to develop how λ-tracking can also be applied to systems in the general form (47). Instead we restrict our attention to a simple example which captures the typical features to show how the global minimum phase condition can be replaced. Consider the following two-dimensional model for chicken manure treatment presented by Hill and Barth (1977): the one stage reaction scheme $S \hookrightarrow X + Q$ is modelled by the system of ordinary nonlinear differential equations:

$$\left.\begin{aligned} \dot{S}(t) &= -k_1 \cdot \mu(S(t)) \cdot X(t) - D(t) \cdot S(t) + D(t) S_{\text{in}} \\ \dot{X}(t) &= \mu(S(t)) \cdot X(t) - D(t) \cdot X(t) \end{aligned}\right\} \tag{49}$$

$S(t)$ denotes the organic concentration [gl^{-1}], $X(t)$ denotes the concentration of methanogenic bacteria [gl^{-1}], S_{in} denotes the organic concentration in the influent [gl^{-1}], Q denotes the biogas production rate (not important for our analysis), $D(t)$ denotes the dilution rate [d^{-1}] at time t [d], and $\mu(S) = (\mu_{\max} \cdot S)/(K_m + S)$ is the specific bacteria growth rate by Michaelis–Menten for some positive constants $\mu_{\max}$, K_m and $k_1 > 0$. Here the dilution rate is considered as input $u(\cdot) \equiv D(\cdot)$ and the organic concentration as output $y(\cdot) \equiv S(\cdot)$.

Note that (49) is not in input affine form. There exist different transformations to convert it into input affine form (e.g. $z := X(S_{\text{in}} - S)^{-1}$, $y := S$ or $\eta := (S_{\text{in}} - S)X^{-1}, y := S$) but the transformations are not globally defined.

We restrict the initial conditions of (47) to lie in the open triangle

$$\Omega := \{(S, X) \mid 0 < S,\ 0 < X,\ S + k_1 X < S_{\text{in}}\}$$

since the polluted organics and bacteria concentrations are positive. Under the assumptions

$$\left.\begin{aligned} &\hat{u}(\cdot)\colon [0,\infty) \to [0,\bar{\Delta}], \qquad \bar{\Delta} > 0 \\ &y_{\text{ref}}(\cdot), n(\cdot) \in \mathcal{W}^{1,\infty} \qquad \text{such that} \\ &y_{\text{ref}}(\cdot), n(\cdot)\colon [0,\infty) \to [\underline{S}, \bar{S}], \qquad \text{where} \quad 0 < \underline{S} \leqslant \bar{S} < S_{\text{in}} \end{aligned}\right\} \tag{50}$$

Ω is invariant under the flow if the feedback

$$u(t) = D(t) = -k(t)[S(t) - y_{\text{ref}}(t) - n(t)] + \hat{u}(t) \tag{51}$$

is applied to (49) and moreover a bound on $S(\cdot)$ can be determined.

Proposition 4. If (50) holds, $k(\cdot)\colon [0,\infty) \to (0,\infty)$ is continuous, monotonically non-decreasing and unbounded, then for each $(S_0, X_0) \in \Omega$ the feedback law (51) for $u(\cdot) \equiv D(\cdot)$ applied to (49) yields a unique solution $(S(\cdot), X(\cdot))\colon [0,\infty) \to \Omega$ and, for some $t' > 0$

$$S(t) < \frac{S_{\text{in}} + \bar{S}}{2} \qquad \text{for all} \qquad t \geqslant t' \tag{52}$$

Proof. To see that $S \equiv 0$ is repelling, note that $S(t') = 0$ for some $t' > 0$ yields

$$\dot{S}(t') = D(t')S_{\text{in}} = [k(t')\,(y_{\text{ref}}(t') + n(t')) + \Delta(t')]\,S_{\text{in}} > k(t')\underline{S}S_{\text{in}} > 0$$

Positivity of $X(t)$ follows from

$$X(t) = \exp\left\{\int_0^t [\mu(S(\tau)) - D(\tau)]\,\mathrm{d}\tau\right\} X_0 > 0$$

To complete the proof of invariance of Ω note that

$$\frac{\mathrm{d}}{\mathrm{d}t}[S(t) + k_1 X(t) - S_{\text{in}}] = -D(t)\,[S(t) + k_1 X(t) - S_{\text{in}}]$$

and hence

$$[S(t) + k_1 X(t) - S_{\text{in}}] = \exp\left\{-\int_0^t D(\tau)\,\mathrm{d}\tau\right\}[S_0 + k_1 X_0 - S_{\text{in}}] < 0$$

It remains to prove (52). Choose $t_1 > 0$ such that

$$-k(t_1)\frac{S_{\text{in}} - \bar{S}}{2} + \bar{\Delta} < 0 \tag{53}$$

We prove that for arbitrary $t_2 > t_1$ there exists some $t_3 > t_2$ such that

$$S(t_3) \leqslant \frac{S_{\text{in}} + \bar{S}}{2} \tag{54}$$

Seeking a contradiction suppose

$$S(t) > \frac{S_{\text{in}} - \bar{S}}{2} \qquad \text{for all} \qquad t \geqslant t_2 \tag{55}$$

Then, by (55), (50), (53) and monotonicity of $t \mapsto k(t)$, we may conclude, for all $t \geqslant t_2$

$$D(t) = -k(t)\,[S(t) - \bar{S}] + \hat{u}(t) < -k(t)\frac{S_{\text{in}} - \bar{S}}{2} + \bar{\Delta} < -k(t_1)\frac{S_{\text{in}} - \bar{S}}{2} + \bar{\Delta} < 0$$

and therefore

$$\dot{S}(t) < D(t)S_{\text{in}} < -\left[k(t_1)\frac{S_{\text{in}} - \bar{S}}{2} - \bar{\Delta}\right]S_{\text{in}} < 0$$

which contradicts (55).

Finally we choose $t_3 > t_1$ such that (54) holds. Then for each $t \geqslant t_3$ such that

$$S(t) = \frac{S_{\text{in}} + \bar{S}}{2}$$

we have

$$\dot{S}(t) < -D(t)\left[\frac{S_{\text{in}} + \bar{S}}{2} - S_{\text{in}}\right] \leqslant -k(t)\left[\frac{S_{\text{in}} + \bar{S}}{2} - \bar{S}\right]\frac{S_{\text{in}} - \bar{S}}{2} < 0.$$

This proves (52) for $t' = t_3$ and completes the proof.

With boundedness of $S(t)$ as given in (52) it is easy to show the High-Gain Property (A2).

Lemma 4 (high-gain lemma for systems with bounded flow). Suppose (50) holds true and $k(\cdot)\colon [0, \infty) \to [0, \infty)$ is continuous, monotonically non-decreasing and unbounded. Then the feedback (51) applied to (49) with $(S_0, X_0) \in \Omega$ yields

$$\lim_{t \to \infty} [S(t) - y_{\text{ref}}(t) - n(t)] = 0$$

Proof. By Proposition 4 the closed-loop system for $e(t) = S(t) - y_{\text{ref}}(t) - n(t)$

$$\dot{e}(t) = -k_1\mu(S(t))X(t) - (\dot{y}_{\text{ref}}(t) + \dot{n}(t)) + [S_{\text{in}} - S(t)]\,[-k(t)e(t) + \tilde{u}(t)]$$

$$\dot{x}(t) = \{\mu(S(t)) - [-k(t)e(t) + \tilde{u}(t)]\}\,X(t)$$

permits a unique solution and we may conclude, for all $t \geqslant t'$

$$\dot{e}(t) \leqslant -\frac{S_{\text{in}} - \bar{S}}{2}k(t)e(t) + M$$

where

$$M := \| -k\mu(S(\cdot))X(\cdot) - \dot{y}_{\mathrm{ref}}(\cdot) - \dot{n}(\cdot) + (S_{\mathrm{in}} - S(\cdot))\hat{u}(\cdot) \|_{L_\infty(0,\infty)}$$

Hence, by Variations-of-Constants, for all $t \geqslant t_0 \geqslant t'$

$$e(t) \leqslant \mathrm{e}^{-\alpha k(t_0)(t-t_0)}\, e(t_0) + \frac{M}{\alpha k(t_0)}$$

where $\alpha := (S_{\mathrm{in}} - \bar{S})/2$. Since $k(t_0)$ may be chosen arbitrarily large, the statement follows and the proof is complete.

We are now in a position to finalize this section by showing that (49) satisfies the assumptions (A1)–(A4), and hence the adaptive strategy (51), (13) can be applied.

Proposition 5. The class of systems (49) satisfying (50) with control (51) is a subset of systems (11), (12) satisfying (A1)–(A4). Hence the control objectives (15) are met if (13), (51) are applied to (49).

Proof. Existence and uniqueness of the solution of the closed-loop system on $[0, \infty)$ is ensured by Proposition 4. (A1) is a consequence of Lemma 4, (A2) and (A4) follow immediately from Proposition 4. Hence, by Theorem 1 the control objectives (15) hold true. This completes the proof.

7 Simulations

In this section, we illustrate that the adaptive control strategy (12), (13) meets the given objectives of λ-tracking with good transient performance. The gain settles at a level only slightly larger than in a non-adaptive context and, most importantly, the terminal gain is the closer to the gain needed in a non-adaptive context the larger r in (13) is. These statements are no longer valid if the sign of the high-frequency gain is unknown; then the simple Byrnes–Willems stabilizer $u(t) = k(t)\cos\sqrt{k(t)}\,y(t), \dot{k}(t) = y(t)^2$ (due to Nussbaum (1983)) depicts a non-convincing transient behaviour, see p. 163 of Ilchmann (1993) for some simulations and comparisons.

Instead of the two-dimensional model (49), we may consider a more detailed five-dimensional model of the anaerobic fermentation process in a continuous stirred tank reactor, in particular the degradation of organic waste by micro-organisms in the absence of oxygen. (See Bastin and Dochain (1990), Bastin and Van Impe (1995) and Stoyanov and Simeonov (1995).) It is based on the three-stage reaction scheme

$$S_0 \hookrightarrow S_1$$

$$S_1 \hookrightarrow X_1 + S_2$$

$$S_2 \hookrightarrow X_2 + Q$$

where S_0 denotes the influent polluting organics concentration $[\mathrm{mg\,l^{-1}}]$, S_1 denotes the substrate concentration for acidogenic bacteria $[\mathrm{mg\,l^{-1}}]$, S_2 denotes the concentration of methanogenic bacteria $[\mathrm{mg\,l^{-1}}]$, Q denotes the biogas production rate $[\mathrm{l\,d^{-1}}]$, and $S = S_0 + S_1 + S_2$ denotes the total organic concentration in the reactor.

Its mathematical model is given by

$$\left.\begin{aligned}
\dot{S}_0(t) &= -bS_0(t)X_1(t) && -[S_0(t)-y_pS_{\mathrm{in}}]\cdot D(t)\\
\dot{X}_1(t) &= \mu_1(S_1(t))X_1(t)-k_1X_1(t) && -X_1(t)\cdot D(t)\\
\dot{S}_1(t) &= bX_1(t)S_0(t)-\frac{\mu_1(S_1(t)X_1(t)}{y_1} && -S_1(t)\cdot D(t)\\
\dot{X}_2(t) &= \mu_2(S_2(t))X_2(t)-k_2X_2(t) && -X_2(t)\cdot D(t)\\
\dot{S}_2(t) &= y_b\mu_1(S_1(t))X_1(t)-\frac{\mu_2(S_2(t))X_2(t)}{y_2} && -S_2(t)\cdot D(t)
\end{aligned}\right\} \tag{56}$$

where S_{in} denotes the organic concentration in the influent, $D(t)$ denotes the dilution rate $[\mathrm{d}^{-1}]$ at time $t\,[\mathrm{d}]$, and $\mu_i(S)=(\mu_{\max,i}\cdot S)/(S+K_{mi})$ for $i=1,2$ denote the Michaelis–Menten growth rates for bacteria.

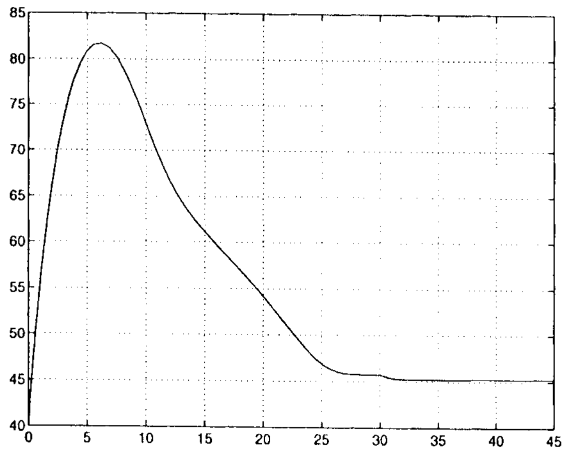

Fig. 2. Output $t\mapsto S(t)$ for λ-tracker (12), (13) with $\gamma=1$, $r=2$, $\lambda=0.5$, $k(30)=0$, $\hat{u}(\cdot)\equiv 0.23$ applied to (56).

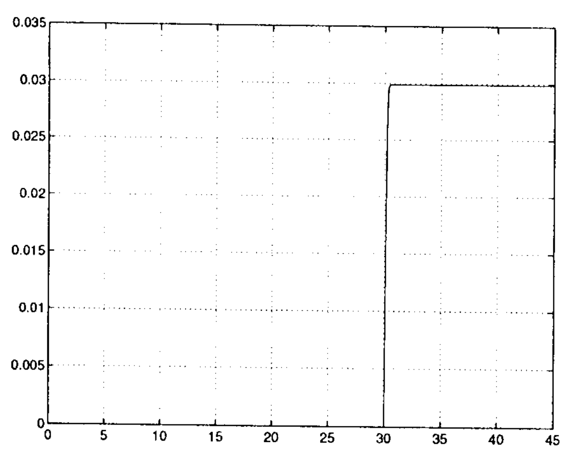

Fig. 3. Gain $t\mapsto k(t)$ of λ-tracker (12), (13) with $\gamma=1$, $r=2$, $\lambda=0.5$, $k(30)=0$, $\hat{u}(\cdot)\equiv 0.23$ applied to (56).

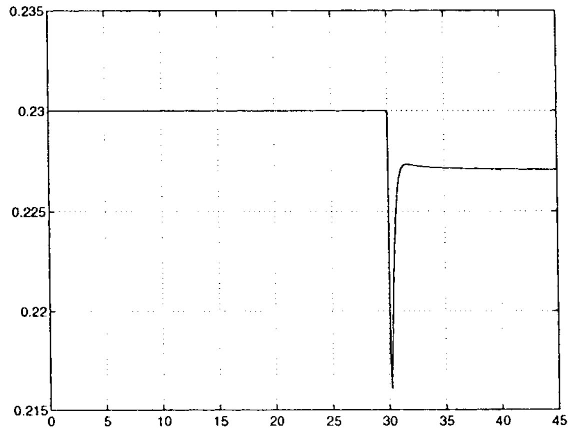

Fig. 4. Input $t \mapsto u(t)$ for λ-tracker (12), (13) with $\gamma = 1$, $r = 2$, $\lambda = 0.5$, $k(30) = 0$, $\hat{u}(\cdot) \equiv 0.23$ applied to (56).

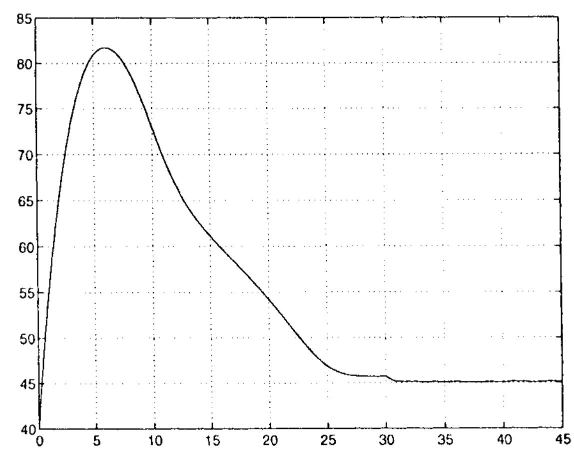

Fig. 5. Output $t \mapsto S(t)$ for λ-tracker (12), (13) with $\gamma = 1$, $r = 2$, $\lambda = 0.3$, $k(30) = 0$, $\hat{u}(\cdot) \equiv 0.23$ and noise generated by (57) applied to (56).

It is easy to show that an analogous result as in Proposition 5 is also valid for (56). This is omitted for brevity.

We simulate the five-dimensional model (56) based on the following experimental data derived by Stoyanov and Simeonov (1995):

$$\mu_{\max,i} = 0.4, \quad k_i = 0.02, \quad K_{mi} = 1 \quad \text{for} \quad i = 1, 2$$
$$b = 1, \quad y_p = 2, \quad y_1 = 0.006, \quad y_2 = 1.1, \quad y_b = 40, \quad S_{\text{in}} = 70\,[\text{mg}\,\text{l}^{-1}]$$

In all of the following simulations, we choose the above parameters and the initial values

$$(S_0(0), X_1(0), S_1(0), X_2(0), S_2(0)) = (40, 0.2, 0, 0.2, 0)$$

the setpoint to be tracked is $y_{\text{ref}}(\cdot) \equiv S_{\text{ref}} = 45$ [mg l^{-1}]. As output of the system, we consider the total organic concentration $y(\cdot) := S(\cdot) = S_1(\cdot) + S_2(\cdot) + S_3(\cdot)$, and the input will be the dilution rate $u(\cdot) := D(\cdot)$.

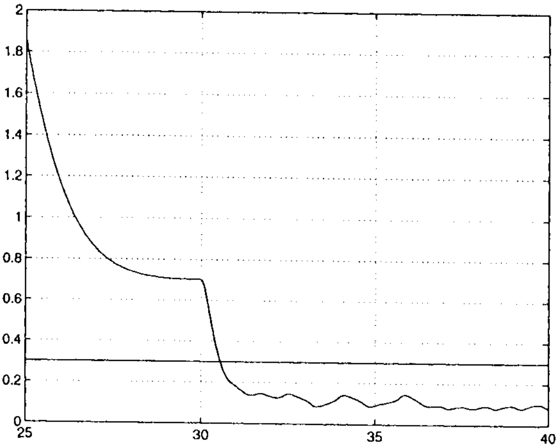

Fig. 6. Output $t \mapsto \mathrm{dist}(|S(t)| - y_{\mathrm{ref}})$ on $[25, 40]$ for λ-tracker (12), (13) with $\gamma = 1$, $r = 2$, $\lambda = 0.3$, $k(30) = 0$, $\hat{u}(\cdot) \equiv 0.23$ and noise generated by (57) applied to (56).

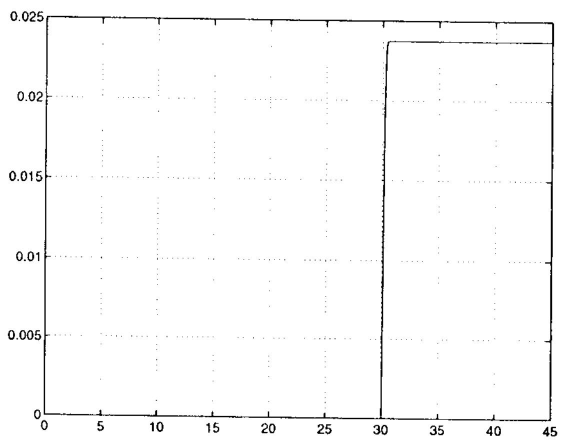

Fig. 7. Gain $t \mapsto k(t)$ of λ-tracker (12), (13) with $\gamma = 1$, $r = 2$, $\lambda = 0.3$, $k(30) = 0$, $\hat{u}(\cdot) \equiv 0.23$ and noise generated by (57) applied to (56).

Simulation 1. We first allow the system to converge to an equilibrium point within 30 days by setting $u(\cdot) \equiv \hat{u}(\cdot) \equiv 0.23$. At time $t \geqslant 30$, the system has settled and we switch on the λ-tracker (12), (13) with design parameters

$$\gamma = 1, \qquad \lambda = 0.5, \qquad r = 2, \qquad k(0) = 0, \qquad \hat{u}(\cdot) \equiv 0.23$$

We do not corrupt the output by noise. Note that the transient behaviour shown in Figs 3–5 is very good. Within a day, the output $S(t)$ is forced without any oscillations into the interval $[44.5, 45.5]$. The output stays in the tolerance band and the gain converges to a finite value, which is not larger than 0.03. See also the good performance of $u(t)$.

Simulation 2. The difference to Simulation 1 is that this time we choose $\lambda = 0.3$ and corrupt the output by some noise to be chosen as $n(t) = \frac{1}{20} q_1(t)$, where $q_1(\cdot)$

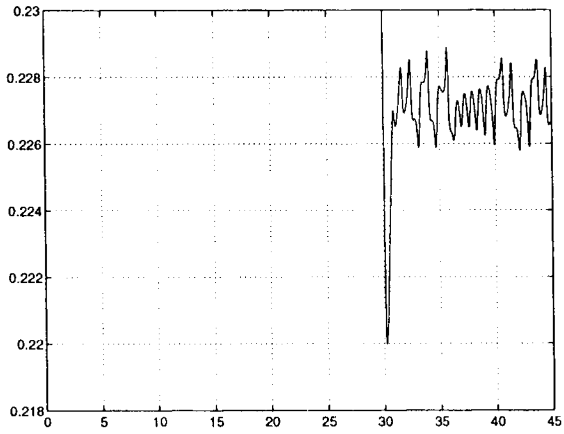

Fig. 8. Input $t \mapsto u(t)$ of λ-tracker (12), (13) with $\gamma = 1$, $r = 2$, $\lambda = 0.3$, $k(30) = 0$, $\hat{u}(\cdot) \equiv 0.23$ and noise generated by (57) applied to (56).

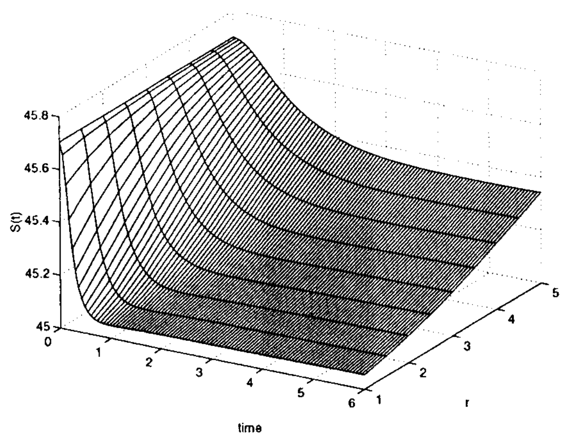

Fig. 9. Output $t \mapsto S(t)$ for λ-tracker (12), (13) with $\gamma = 1$, $\lambda = 0.3$, $k(30) = 0$, $\hat{u}(\cdot) \equiv 0.23$ applied to (56), $r = 1, \ldots, 5$.

denotes the first component of the Lorenz equation:

$$\left.\begin{aligned} \dot{q}_1(t) &= 10[q_2(t) - q_1(t)], & q_1(0) &= 1 \\ \dot{q}_2(t) &= 28q_1(t) - q_2(t) - q_1(t)q_3(t), & q_2(0) &= 0 \\ \dot{q}_3(t) &= q_1(t)q_2(t) - \frac{8}{3}q_3(t), & q_3(0) &= 3 \end{aligned}\right\} \tag{57}$$

Sparrow (1982) shows that the parameters in (57) ensure chaotic and bounded behaviour of $q_i(\cdot)$ with bounded derivative, $i = 1, 2, 3$. In this case, $|n(t)| < 0.1$.

We obtain the same qualitative results, see Figs 6–9, and we have added a cutting of the output dynamics around the point $t = 30$, where the regulator is switched on, see Fig. 7.

Simulation 3. In the last simulation we illustrate the effect of varying the parameter r in the gain adaptation (13). For simplicity, we consider the noise free case and

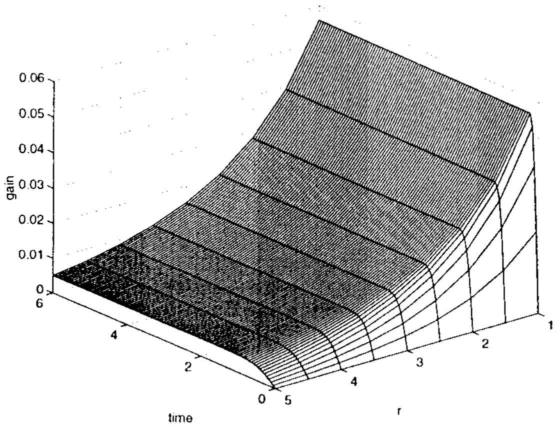

Fig. 10. Gain $t \mapsto k(t)$ of λ-tracker (12), (13) with $\gamma = 1$, $\lambda = 0.3$, $k(30) = 0$, $\hat{u}(\cdot) \equiv 0.23$ applied to (56), $r = 1, \ldots, 5$.

put $\lambda = 0.3$. We depict the output $S(t)$ and the gain $k(t)$ only for 6 days after the regulator has been switched on at $t = 30$. See Figs 7 and 8. r is varied for 1–5 with stepsize 0.5. The larger we choose r to be, the faster the gain dynamics are. For $r = 1$, the dynamics of $k(t)$ are slow: when the gain is large enough so that the output is forced towards the λ-ball, $k(t)$ still increases until finally it is switched off when $S(t)$ enters the λ-ball. This overshoot of the gain becomes much smaller when a larger r is applied. See Fig. 10. One might argue that a larger terminal gain forces the output closer to the reference setpoint, however this is not the control objective. Note that for $r = 5$ the terminal gain is much smaller than for $r = 1$, on the price that reaching the tolerance band takes longer.

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